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Poisson-Lie T-plurality of three-dimensional conformally invariant sigma models II: nondiagonal metrics and dilaton puzzle

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Abstract: We look for 3-dimensional Poisson-Lie dualizable sigma models that satisfy the vanishing β-function equations with constant dilaton field. Using the Poisson-Lie T-plurality we then construct 3-dimensional sigma models that correspond to various decompositions of Drinfeld double. Models with nontrivial dilaton field may appear. It turns out that for "traceless" dual algebras they satisfy the vanishing β-function equations as well. In certain cases the dilaton cannot be defined in some of the dual models. We provide an explanation why this happens and give criteria predicting when it happens.

Keywords: Sigma Models, String Duality.

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Contents

1. Introduction **1**

2. Construction of Poisson-Lie T-dual sigma models **2**

3. The Poisson-Lie T-plurality **3**

4. Dilaton puzzle **5**

5. Sigma models on solvable three-dimensional groups **7**
5.1 Sigma models on DD11 starting from $(5|1)$ 8
5.2 Sigma models on DD11 starting from $(6_0|1)$ 11
5.3 Sigma models on DD12 starting from $(4|1)$ 13
5.4 Sigma models on DD13 starting from $(3|1)$ 15
5.5 Sigma models on DD15 starting from $(7_0|1)$ 15
5.6 Sigma models on DD19 starting from $(2|1)$ 16
5.7 Sigma models on DD22 17

6. Conclusions **17**

A. Drinfeld doubles **19**
A.1 Structure of DD11 20
A.2 Structure of DD12 21
A.3 Structure of DD13 21
A.4 Structure of DD15 21
A.5 Structure of DD19 22
A.6 Structure of DD22 22

1. Introduction

As it was shown in [1], Poisson-Lie T-dual sigma models are given by a Manin triple, i.e. a decomposition of a classical Drinfeld double, and by an invertible constant matrix E_0. Construction of the dual models is described e.g. in [1, 2, 3]. Examples of dual models are given e.g. in [4]–[9].

The fact that for a given Drinfeld double several Manin triples may exist leads to the notion of Poisson-Lie T-plurality [10]. In our previous paper [11] we presented the Poisson-Lie T-plurality for a rather special class of sigma models. We looked for the conformally invariant models with the diagonal matrix E_0 and vanishing dilaton field. We obtained models for the Manin triples $(6_0|1), (7_0|1)$ (for the notation see appendix A) and then investigated their associated models.

In this paper we follow a more systematic approach, namely, we do not restrict ourselves to the diagonal E_0's.[1] That provide us with much larger set of the three-dimensional conformally invariant sigma models. We present general forms of the conformally invariant sigma models with vanishing (more precisely, constant) dilaton field on the Manin triples $(1|1), (2|1), (3|1), (4|1), (5|1), (6_0|1), (7_0|1)$ and using the Poisson-Lie T-plurality we construct the conformally invariant sigma models on the other Manin triples in the corresponding Drinfeld doubles.

To set our notation let us briefly review the construction of the Poisson-Lie T-plural sigma models [10] in the next two sections. In section 4 we discuss the problems with transformations of dilaton field. Obtained three-dimensional conformally invariant sigma models are given in section 5.

2. Construction of Poisson-Lie T-dual sigma models

The Poisson-Lie T-dual sigma models are constructed by means of Drinfeld doubles. The Drinfeld double D is defined as a connected Lie group such that its Lie algebra $\mathcal{D}$ can be decomposed into a pair of subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$ maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form $\langle\, .\,,.\, \rangle$ on $\mathcal{D}$. The dimensions of the subalgebras are equal and due to the ad-invariance of $\langle\, .\,,.\, \rangle$ the algebraic structure of $\mathcal{D}$ is determined by the structure of the maximally isotropic subalgebras (see appendix A).

The lagrangian of the dualizable sigma models

$$L = F_{ij}(y)\partial_- y^i \partial_+ y^j\,, \qquad i,j = 1,\ldots,n = \dim \mathcal{G} \tag{2.1}$$

can be rewritten in terms of right-invariant fields as[2]

$$L = E_{ab}(g)(\partial_- g g^{-1})^a (\partial_+ g g^{-1})^b,\ g \in G \tag{2.2}$$

where

$$E(g) = (E_0^{-1} + \Pi(g))^{-1}\,, \qquad \Pi(g) = b(g)a(g)^{-1} = -\Pi(g)^t\,. \tag{2.3}$$

and $a(g), b(g), d(g)$ are $n \times n$ submatrices of the adjoint representation of the group G on $\mathcal{D}$ in the basis $(X_i, \tilde{X}^j)$[3]

$$Ad(g)^t = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}, \tag{2.4}$$

$$a(g)^{-1} = d(g)^t\,, \qquad b(g)^t a(g) = -a(g)^t b(g)\,. \tag{2.5}$$

[1]This explains the subtitle of the current paper. It is a bit imprecise since even diagonal matrices E_0 in most cases give nondiagonal metrics. Because E_0 coincides with the sum of metric and torsion potential at the group unit, a completely precise subtitle would be "metrics nondiagonal at the group unit".

[2]G ($\tilde{G}$) is the subgroup of D whose Lie algebra is $\mathcal{G}$ ($\tilde{\mathcal{G}}$). All constructions are in general permissible only locally, in a vicinity of the group unit.

[3]t denotes transposition.

JHEP10(2004)045

It means that

$$F_{ij}(y) = e_i^a(g(y))E_{ab}(g(y))e_j^b(g(y)) \tag{2.6}$$

where e_i^a are components of right-invariant forms (vielbeins) $e_i^a(g) = \big((\mathrm{d}g)_i \cdot g^{-1}\big)^a$ and y^i are local coordinates of $g \in G$.

By a modification of this procedure one can construct dual models even for noninvertible matrices E_0 (see [13]) but we shall not consider such models here, because then the dual model is not of the form (2.2).

The covariant tensor field F on G is thus determined by the decomposition $\mathcal{D} = (\mathcal{G}|\tilde{\mathcal{G}})$ and by the matrix E_0. It can be understood as a sum of the metric and the torsion potential defining the geometric properties of the manifold G. Necessary condition for invertibility of the metric of sigma models is

$$\det(E_0 + E_0^t) \neq 0\,. \tag{2.7}$$

It turns out that usually this condition is also sufficient.

The ultraviolate finiteness of the quantum version of the model is guaranteed by the conformal invariance of the model. To achieve this invariance at the one-loop level we must add another term containing the so-called dilaton field to the lagrangian. The dilaton field Φ can be understood as an additional function on G that defines the quantum nonlinear sigma model. The conformal invariance of the model is guaranteed by vanishing of the so-called β-function. At the one-loop level the equations for vanishing of the β-function read

$$0 = R_{ij} - \bigtriangledown_i \bigtriangledown_j \Phi - \frac{1}{4}H_{imn}H_j^{mn}\,, \tag{2.8}$$

$$0 = \bigtriangledown^k \Phi H_{kij} + \bigtriangledown^k H_{kij}\,, \tag{2.9}$$

$$0 = R - 2\bigtriangledown_k \bigtriangledown^k \Phi - \bigtriangledown_k \Phi \bigtriangledown^k \Phi - \frac{1}{12}H_{kmn}H^{kmn} \tag{2.10}$$

where the covariant derivatives $\bigtriangledown_k$, Ricci tensor R_{ij} and Gauss curvature R are calculated from the metric

$$G_{ij} = \frac{1}{2}(F_{ij} + F_{ji}) \tag{2.11}$$

that is also used for lowering and raising indices, and the torsion is

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} \tag{2.12}$$

where

$$B_{ij} = \frac{1}{2}(F_{ij} - F_{ji})\,. \tag{2.13}$$

3. The Poisson-Lie T-plurality

The possibility to decompose some Drinfeld doubles into more than two Manin triples[4] enables us to construct more than two equivalent sigma models and this property is called Poisson-Lie T-plurality [10]. Let $\{X_j, \tilde{X}^k\},\ j,k \in \{1,\ldots,n\}$ be generators of Lie subalge-

[4]Two decompositions always exist, $(\mathcal{G}|\tilde{\mathcal{G}})$ and $(\tilde{\mathcal{G}}|\mathcal{G})$.

JHEP10(2004)045

bras $\mathcal{G}, \tilde{\mathcal{G}}$ of a Manin triple associated with the lagrangian (2.2) and $\{U_j, \tilde{U}^k\}$ are generators of some other Manin triple $(\mathcal{G}_u, \tilde{\mathcal{G}}_u)$ in the same Drinfeld double related by the $2n \times 2n$ transformation matrix:

$$\begin{pmatrix} \vec{X} \\ \vec{\tilde{X}} \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \vec{U} \\ \vec{\tilde{U}} \end{pmatrix}, \tag{3.1}$$

where

$$\vec{X} = (X_1, \ldots, X_n)^t, \ldots, \ \vec{\tilde{U}} = (\tilde{U}^1, \ldots, \tilde{U}^n)^t .$$

The transformed model is then given by the lagrangian of the same form as (2.2) but with $E(g)$ replaced by

$$E_u(g_u) = M(N + \Pi_u\, M)^{-1} = (E_{0u}^{-1} + \Pi_u)^{-1} \tag{3.2}$$

where

$$M = S^t E_0 - Q^t\,, \qquad N = P^t - R^t E_0\,, \qquad E_{0u} = MN^{-1} \tag{3.3}$$

and Π_u is calculated by (2.3) from the adjoint representation of the group G_u generated by $\{U_j\}$. The transformation of E_0 corresponds to the invariance of $\mathcal{E}^+ = \mathrm{span}\{X_j + (E_0)_{jk}\tilde{X}^k\} = \mathrm{span}\{U_j + (E_{0u})_{jk}\tilde{U}^k\}$. Note that for $P = S = 0$, $Q = R = \mathbf{1}$ we get the dual model with $E_{0u} = E_0^{-1}$, corresponding to the interchange $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$ so that the duality transformation is a special case of the plurality transformation (3.1)–(3.3).

In quantum theory the duality or plurality transformation must be supplemented by a correction that comes from integrating out the fields on the dual group $\tilde{G}$ in path integral formulation. In some cases it can be absorbed at the 1-loop level into the transformation of the dilaton field Φ . The transformation of the tensor F that follows from (3.2) must be accompanied by the transformation of the dilaton [10]

$$\Phi_u = \Phi + \ln|\,\mathrm{Det}(N + \Pi_u M)| - \ln|\,\mathrm{Det}(\mathbf{1} + \Pi E_0)| + \ln|\,\mathrm{Det}\, a_u| - \ln|\,\mathrm{Det}\, a| \tag{3.4}$$

where Π_u, a_u are calculated as in (2.4), (2.3) but from the adjoint representation of the group G_u. Unfortunately this transformation of the dilaton field cannot be applied in general. This problem will be discussed in the next section.

Beside the transformations that follow from (3.1) we can rescale the matrix E_0 and thus also $F(y)$ by a constant factor. For that one should notice that although there are nonisomorphic Manin triples whose commutation relations differ just by an overall multiplication constant κ in all the commutators in the second subalgebra $\tilde{\mathcal{G}}$, such Manin triples lead to equivalent models. The reason is that such rescaling leads to

$$a(g) \to a(g)\,, \qquad b(g) \to \kappa\, b(g)\,, \qquad d(g) \to d(g)$$

and consequently to rescaling of the metric of such model

$$E_0 \to \frac{E_0}{\kappa}\,, \qquad F \to \frac{F}{\kappa}\,. \tag{3.5}$$

It is easy to see that such transformation converts one solution of the vanishing β-function equations (2.8)–(2.10) into another one, since all terms in each of the equations scale in the same way. In fact this transformation corresponds to the rescaling of string tension if the model (2.2) is interpreted in the context of string theory.

4. Dilaton puzzle

For further reference, it is convenient to introduce in accordance with [10],

$$\Phi^{(0)} = \Phi - \ln|\,\mathrm{Det}(\mathbf{1} + \Pi E_0)| - \ln|\,\mathrm{Det}\ a|\,. \tag{4.1}$$

One may introduce also

$$\Phi_u^{(0)} = \Phi_u - \ln|\,\mathrm{Det}(\mathbf{1} + \Pi_u M N^{-1})| - \ln|\,\mathrm{Det}\ a_u| = \Phi^{(0)} + \ln|\,\mathrm{Det}\,N| \tag{4.2}$$

showing that although $\Phi^{(0)}$ can be considered a function on the whole Drinfeld double by trivial extension

$$\Phi^{(0)}(g \cdot \tilde{g}) = \Phi^{(0)}(g)\,, \tag{4.3}$$

it is not defined unambiguously by the Drinfeld double and the subspace $\mathcal{E}^+$; the Manin triple must be also specified. Starting from equivalent models on different Manin triples one obtains $\Phi^{(0)}$s differing by an additive function of eventual spectator fields. (In our paper we shall not consider spectators and those $\Phi^{(0)}$s may differ by a constant.) On the other hand, from (4.2) follows that the dependence of $\Phi^{(0)}$ on the coordinates of the Drinfeld double is the same for any choice of the Manin triple. This shall prove to be important in future considerations.

As shown in [10, 11], in all known examples of Poisson-Lie T-plural models the conformal invariance is preserved under the T-plurality transformations (3.2), (3.4) provided the new dilaton Φ_u is well defined. That it may not be the case was observed in [10] in an example with a spectator field present and in this paper we shall present also other examples without spectators. In both cases new dilaton cannot be defined because it depends on the coordinates on the subgroup $\tilde{G}_u$ which were presumably integrated out in path integral. The reason for this was not understood. In the following we shall give a criterion when the new dilaton Φ_u exists and also an explanation from where the trouble arises.

An obvious necessary and sufficient condition for the existence of well-defined Φ_u is that Φ_u doesn't depend on the coordinates on the subgroup $\tilde{G}_u$. In the light of the definitions (4.1), (4.2) of $\Phi^{(0)}$ and $\Phi_u^{(0)}$ this is equivalent to the condition that $\Phi_u^{(0)}$ doesn't depend on the element $\tilde{g}_u$ of $\tilde{G}_u$ in the decomposition[5]

$$l = g_u \cdot \tilde{g}_u\,, \qquad g_u \in G_u\,, \quad \tilde{g}_u \in \tilde{G}_u$$

and consequently also that $\Phi^{(0)}$ doesn't depend on $\tilde{g}_u$. Introducing in a vicinity of the group unit a parametrization of elements of the subgroup $\tilde{G}_u$ by an exponential[6] we arrive at the following necessary and sufficient criterion for the existence of the new dilaton Φ_u.

Theorem 1. *The dilaton (3.4) for the model defined on the group G_u exists if and only if*

$$\tilde{U}\Phi^{(0)}(g\cdot\tilde{g}) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi^{(0)}\left(g\cdot\tilde{g}\cdot\exp(t\tilde{U})\right)|_{t=0} = 0\,, \qquad \forall g \in G_u\,, \quad \forall \tilde{g} \in \tilde{G}_u\,, \quad \forall \tilde{U} \in \tilde{\mathcal{G}}_u\,,$$

where $\tilde{U} \in \tilde{\mathcal{G}}_u$ is extended as a left-invariant vector field on D.

[5] Recall (4.3)

[6] and recalling that right action corresponds to left-invariant vector fields

This criterion has an obvious computational disadvantage — although one can rather easily express $U, \tilde{U}$ in terms of basis elements $X_j, \tilde{X}^j$ of $\mathcal{G}, \tilde{\mathcal{G}}$ (the transformation between subalgebras is known, see (3.1)), we also need to rewrite $g_u \cdot \tilde{g}_u(t)$, $g_u \in G_u$, $\tilde{g}_u(t) \in \tilde{G}_u$ as $h(t) \cdot \tilde{h}(t)$, $h(t) \in G$, $\tilde{h}(t) \in \tilde{G}$. Working in the vicinity of the group unit e this can be accomplished using representation of group elements by exponentials and employing Campbell-Baker-Hausdorff formula but such computation can be quite involved. For applications it seems to be much easier to use a weaker necessary condition.

Theorem 2. *A necessary condition for the existence of the dilaton (3.4) for the model defined on the group G_u is*

$$\tilde{U}\Phi^{(0)}(e) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi^{(0)}(\exp(t\tilde{U}))\Big|_{t=0} = 0\,, \qquad \forall \tilde{U} \in \tilde{\mathcal{G}}_u\,.$$

In practice it means to check the matrix R from (3.1) because[7]

$$\begin{pmatrix} \vec{U} \\ \vec{\tilde{U}} \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{-1} \begin{pmatrix} \vec{X} \\ \vec{\tilde{X}} \end{pmatrix} = \begin{pmatrix} S^t & Q^t \\ R^t & P^t \end{pmatrix} \begin{pmatrix} \vec{X} \\ \vec{\tilde{X}} \end{pmatrix} \tag{4.4}$$

so that

$$\tilde{U}^k \Phi^{(0)}(e) = R^{jk} X_j \Phi^{(0)}(e) = R^{jk} \frac{\partial \Phi^{(0)}(y)}{\partial y_j}|_{y=0} \tag{4.5}$$

due to the convention (5.1).

In all examples known to us the necessary condition proved to be also sufficient — the new dilaton is in such cases constructed explicitly. Special cases of matrices from (4.4) can be found in [12] but here we work with their general forms.

It is rather surprising that in all further examples it is either possible to find new dilatons for transformations to all isomorphic Manin triples (i.e. isomorphic subalgebras immersed in different ways into $\mathcal{D}$) or it is impossible for all of them. This coincidence seems to indicate some nontrivial relation between dilatons as solutions of (2.8)–(2.10) for the metric and the torsion given by (2.2) and the Poisson-Lie T-plurality transformations.

Also it is clear that if $\Phi^{(0)}$ is a nontrivial (i.e. nonconstant) function on G then it is impossible to perform Poisson-Lie T-plurality transformation to the dual group $\tilde{G}$, i.e. to the Manin triple obtained by just interchanging the subalgebras in the original one (since in this case $\tilde{\mathcal{G}}_u = \mathcal{G}$ and exists $X \in \mathcal{G}$ such that $X\Phi^{(0)} \neq 0$). This casts additional doubts on the suitability of the original term "duality" in quantum theory.

Another strange discovery is that in some cases when one starts from conformally invariant model and the new dilaton (3.4) doesn't exist, there is nevertheless a function Φ'_u on G_u such that the 1-loop β-function is vanishing also in the new model. Whether it is possible to somehow relate such models is unclear (new dilatons were in those cases guessed and no relations to the original ones are known).

Also the origin of the dilaton puzzle now becomes clear. When the transformation of the dilaton was derived in [10] using path integral (see section 3 therein), the crucial step

[7] Note that the second equality follows from the fact that transformation (3.1) preserves the bilinear form $\langle .,. \rangle$, i.e. the relations (A.1).

was integrating out the dependence on the auxiliary group $\tilde{G}$. This was performed using a change of variables from elements of the group $\tilde{G}$ to components of invariant 1-forms $\tilde{g}^{-1}\partial\tilde{g}$ and integrating functional δ-function. Terms that appeared in the regularization of a certain functional determinant were then absorbed into a shift of a "naive" or "bare" dilaton $\Phi^{(0)}$ giving the relation between $\Phi^{(0)}$ and the "true" dilaton Φ as given in (4.1). During this computation it was tacitly assumed that $\Phi^{(0)}$ is not involved in the integration over $\tilde{G}$, i.e. that it doesn't depend on it.

It becomes clear that if $\Phi^{(0)}$ depends on $\tilde{G}$ the integration cannot be performed in a similar manner — inverting the relation between invariant 1-forms $\tilde{g}^{-1}\partial\tilde{g}$ and group elements $\tilde{g}$ gives rise to path-ordered exponentials, i.e. nonlocal terms, and the resulting object can be hardly interpreted as a dilaton. It is an open question whether this obstacle can be somehow circumvented or reinterpreted.

5. Sigma models on solvable three-dimensional groups

It follows from the construction of classical dualizable models that they are given by a Manin triple $(\mathcal{G}|\tilde{\mathcal{G}})$ and $n \times n$ invertible matrices E_0 that satisfy (2.7). The quantum models include the dilaton field and require moreover that the equations (2.8)–(2.10) hold. They are equations for the tensor field F (given by the Manin triple and E_0) and the dilaton field Φ. We shall solve them for Manin triples $(\mathcal{G}|\tilde{\mathcal{G}})$ where $\mathcal{G}$ are solvable three dimensional Lie algebras without a parameter[8] in the Bianchi classification (see [17] or e.g. [18]) and $\tilde{\mathcal{G}}$ is three dimensional abelian Lie algebra (i.e. for the triples (1|1), (2|1), (3|1), (4|1), (5|1), $(6_0|1)$, $(7_0|1)$). Moreover we shall assume that $\Phi = const.$ Such solutions exist only for special forms of E_0 given below. Afterwards we shall investigate the sigma models that can be obtained by the transformation of Drinfeld doubles (3.1). We shall omit the models corresponding to the Manin triples where the subalgebra $\tilde{\mathcal{G}}$ is not traceless, i.e. $\tilde{f}_{ij}{}^{j} \neq 0$, because it is known that these quantum sigma models exhibit the so-called gravitational anomaly [14]–[16]. It means that we will investigate the following Drinfeld doubles and decompositions (for the notation see appendix A or [12]):

- DD11: $(5|1) \cong (6_0|1) \cong (1|6_0) \cong (5.ii|6_0) \cong (5|2.i)$,
- DD12: $(4|1) \cong (6_0|2) \cong (2|6_0) \cong (6_0|4.ii) \cong (4|2.i) \cong (4|2.ii)$,
- DD13: $(3|1) \cong (3|2)$,
- DD15: $(7_0|1) \cong (1|7_0)$,
- DD19: $(2|1) \cong (1|2)$.
- DD22: $(1|1)$.

[8] Presently we are not able to perform similar analysis for the Bianchi algebras $\mathbf{6_a}, \mathbf{7_a}$ because of computational difficulties.

Often we do not display the tensor fields F_{ij} because they are usually too complicated (for examples see [11]) but only the matrices $E_0 = F(e)$ from which the tensor fields can be reconstructed by (2.3) and (2.6).

As all groups we are dealing with in the following are solvable we can use the parametrization of $g \in G$ in the form

$$g(y) = \exp(y_1 X_1)\exp(y_2 X_2)\exp(y_3 X_3) \tag{5.1}$$

where y_j are coordinates on the group manifold and X_j are the group generators whose commutation relations are given in the appendix A. We shall not distinguish between y corresponding to different groups by further indices, the group follows from the context. If we need to consider coordinates on different groups in one expression, we denote coordinates on the second group by x. We assume $\epsilon = \pm 1$ in the following.

5.1 Sigma models on DD11 starting from $(5|1)$

Constant dilaton exists for the decomposition $(5|1)$ and

$$E_0 = \begin{pmatrix} p & u & v \\ w & q & g \\ z & g & r \end{pmatrix}, \qquad qr = g^2\,. \tag{5.2}$$

More precisely, the dilaton $\Phi = C \in \mathbb{R}$ and the tensors

$$F_{ij}(y) = \begin{pmatrix} p & u\,\mathrm{e}^{-y_1} & v\,\mathrm{e}^{-y_1} \\ w\,\mathrm{e}^{-y_1} & q\,\mathrm{e}^{-2y_1} & g\,\mathrm{e}^{-2y_1} \\ z\,\mathrm{e}^{-y_1} & g\,\mathrm{e}^{-2y_1} & r\,\mathrm{e}^{-2y_1} \end{pmatrix}, \qquad qr = g^2 \tag{5.3}$$

constructed by (2.6) from $(5|1)$ and (5.2), satisfy the β-function equations (2.8)–(2.10).

The metrics are invertible if and only if $g(v+z) \neq r(u+w)$ for $r \neq 0$ and $q(v+z) \neq 0$ for $r = 0$. The corresponding sigma models are torsionless (i.e. $H_{ijk} = 0$) and their metrics are flat (i.e. their Riemann tensors vanish). (5.2) is the general form of E_0 for which a solution of the β-function equations with constant dilaton exists.[9]

All models given above can be obtained from those with

$$E_0 = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{5.4}$$

by plurality transformations (3.1)–(3.3) that leave the structure constants of $(5|1)$ invariant and we can get rid off the sign by the scaling transformation (3.5).

Besides that by other plurality transformations we can get models corresponding to other decompositions of the DD11. The dilaton field for these models can be obtained from (3.4). However, before using it we should check whether this formula provides us

[9] All calculations done in this section were performed by virtue of Mathematica and Maple so that similar propositions depend on their capability to find all solutions. On the other hand these programs were applied simultaneously and independently so the results seem to be rather reliable.

with a function independent of the coordinates of the subgroup $\tilde{G}_u$. As mentioned in section 4 it is rather difficult, nevertheless, we can check at least the necessary condition given by the Theorem 2. For that we shall need the value of $\Phi^{(0)}$ introduced in section 4. From (4.1) we get

$$\Phi^{(0)}(y) = C - 2y_1 \tag{5.5}$$

so that the condition that follows from (4.5) now reads

$$-2R^{1k} = 0\,, \qquad k = 1,2,3 \tag{5.6}$$

and by inspection of the transformation matrices we can find if it is satisfied. The transformation $(y,\tilde{y}) \to (x,\tilde{x})$ of coordinates on D then can be derived by decomposing an element of D in different ways, i.e.

$$l = g(y)\tilde{g}(\tilde{y}) = g_u(x)\tilde{g}_u(\tilde{x})\,. \tag{5.7}$$

Models different from those above and given by decompositions of the DD11 with the "traceless" second factor correspond to the Manin triples $(5|2.i)$, $(6_0|1)$, $(1|6_0)$ and $(5.ii|6_0)$. Let us investigate these possibilities.

- Decomposition $(5|2.i)$. There are six matrices with up to eight free parameters transforming $(5|1)$ to $(5|2.i)$ by (3.1). Using (3.3) we find that all of them lead to models given by

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & Q & -1-\epsilon \\ X & 1+\epsilon & 0 \end{pmatrix} \tag{5.8}$$

 or

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & (G+1+\epsilon)^2/J & G \\ X & G+2\epsilon+2 & J \end{pmatrix}. \tag{5.9}$$

 The metrics are invertible if and only if $Q(V+X) \neq 0$ and $(2\epsilon+2+G)(V+X) \neq J(U+Y)$ respectively. The corresponding sigma model are torsionless and their metrics are flat.

 By inspection of the transformation matrices one finds that for all of them $R^{1k} = 0$ and from (5.7) one gets $y_1 = -\epsilon x_1$. Using (3.4) one finally finds that the transformed dilaton Φ_u is constant so that the β-function equations (2.8)–(2.10) are again satisfied.

- Decomposition $(6_0|1)$. There are two matrices transforming $(5|1)$ to $(6_0|1)$. They produce models given by

$$\tilde{E}_0 = \begin{pmatrix} Q & 2\epsilon Q - Y & V \\ Y & Q & G \\ X & H & J \end{pmatrix}. \tag{5.10}$$

 The metrics are invertible if and only if $Q(G+H-\epsilon(V+X)) \neq 0$. The corresponding sigma models are torsionless but their metric is not flat. Neither Riemann nor Ricci tensors vanish. Gauss curvature is zero. Similarly as in the previous case one can check that the necessary condition $-2R^{1k} = 0$ for application of the dilaton formula (3.4) is satisfied and the transformation of the coordinates is $y_1 = -\epsilon x_3$. The β-function equations (2.8)–(2.10) are satisfied for the dilaton field

$$\Phi(x) = C + 2\epsilon x_3 \tag{5.11}$$

obtained from (3.4).

- Decompositions $(1|6_0)$. Transformations (3.1)–(3.3) give models with

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & Q & G \\ X & H & J \end{pmatrix} \tag{5.12}$$

where the elements satisfy

$$\begin{aligned} PJ - VX - QJ + GH &= 0\,, \\ GH - PJ - QJ + VX &= \epsilon(HV - UJ + GX - JY)\,. \end{aligned} \tag{5.13}$$

If $J = 0$ then

$$\tilde{E}_0 = \begin{pmatrix} P & U & \epsilon G \\ Y & Q & G \\ \epsilon H & H & 0 \end{pmatrix} \tag{5.14}$$

The corresponding sigma models are torsionless and their metrics are flat.

If $J \neq 0$ then

$$\tilde{E}_0 = \begin{pmatrix} P & [2\epsilon(PJ - VX) + HV + GX]/J - Y & V \\ Y & P + (GH - VX)/J & G \\ X & H & J \end{pmatrix}. \tag{5.15}$$

The corresponding sigma models have nontrivial torsion and their metrics are not flat. The form of dilaton field is not known because $R^{13} = \pm 1$ so that the necessary condition for existence of the dilaton independent of $\tilde{x}$ is violated and the formula (3.4) is not applicable.

- Decomposition $(5.ii|6_0)$. Transformations (3.1)–(3.3) give models with $\tilde{E}_0$ given by (5.12) where the elements satisfy

$$PJ - (V-1)(X+1) - QJ + (G-1)(H+1) = 0\,, \tag{5.16}$$

$$\begin{aligned} &2(\epsilon - 1) + PJ - (V - 1 + \epsilon)(X + 1 - \epsilon) + QJ - \\ &-(G - 1 + \epsilon)(H + 1 - \epsilon) + \epsilon(HV - UJ - JY + GX) = 0\,. \end{aligned} \tag{5.17}$$

If $J = 0$ then

$$\tilde{E}_0 = \begin{pmatrix} P & U & \epsilon(G-1)+1 \\ Y & Q & G \\ \epsilon(X+1)-1 & H & 0 \end{pmatrix}. \tag{5.18}$$

The corresponding sigma models are torsionless and their metrics are flat.

If $J \neq 0$ then

$$Q = P + \frac{G - H + GH - V + X - VX}{J}\,, \tag{5.19}$$

$$U = 2\epsilon P - Y + \frac{2(\epsilon-1) + G - H + HV + (2\epsilon-1)(X-V) + GX - 2\epsilon VX}{J}\,. \tag{5.20}$$

The corresponding sigma models have nontrivial torsion and their metrics are not flat. The form of the dilaton field is not known because $R^{13} = \pm 1$ so that the formula (3.4) is not applicable.

5.2 Sigma models on DD11 starting from $(6_0|1)$

Models on the decomposition[10] $(6_0|1)$ allow constant dilaton for

$$E_0 = \begin{pmatrix} p & u & v \\ -u & -p & g \\ z & h & r \end{pmatrix} \tag{5.21}$$

i.e. for the tensor

$$F_{ij}(y) = \begin{pmatrix} p & u & v + u\,y_1 + p\,y_2 \\ -u & -p & g - p\,y_1 - u\,y_2 \\ z - u\,y_1 + p\,y_2 & h - p\,y_1 + u\,y_2 & r(y) \end{pmatrix} \tag{5.22}$$

where

$$r(y) = r + (g+h)\,y_1 + (v+z)\,y_2 + p\,({y_2}^2 - {y_1}^2)\,.$$

The metric is invertible if and only if $p(4pr + (g+h)^2 - (v+z)^2) \neq 0$. The corresponding sigma models are torsionless and their metrics are flat.

The function $\Phi^{(0)}$ is constant in this case so that we have no problems with its possible dependence on coordinates of auxiliary groups $\tilde{G}_u$ and the formula (3.4) is valid for any decomposition.

All these models can be obtained by a plurality transformation from those with

$$E_0 = \kappa \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.23}$$

and we can get rid off the overall constant $\kappa \in \mathbb{R}\backslash\{0\}$ by the scaling transformation mentioned at the end of section 3.

Beside that by the plurality transformations (3.1) we get the sigma models obtained from

- Decomposition $(1|6_0)$ and

$$\tilde{E}_0 = \begin{pmatrix} P & (HV + GX - JY)/J & V \\ Y & (GH - PJ + VX)/J & G \\ X & H & J \end{pmatrix} \tag{5.24}$$

[10] This set of models was already investigated in [11], although not all possible forms of $\tilde{E}_0$ were explicitly presented for each Manin triple. On the other hand, in [11] formulae for the metric, its Gauss curvature and singularities are usually given in full detail.

JHEP10(2004)045

or

$$\tilde{E}_0 = \begin{pmatrix} P & U & -\epsilon G \\ Y & Q & G \\ \epsilon H & H & 0 \end{pmatrix}. \tag{5.25}$$

In the first case the corresponding sigma models have nontrivial torsion and their metrics are not flat. The dilaton field obtained from (3.4) is

$$\Phi(x) = \ln\left|1 + (G-H)x_1 + (V-X)x_2 + (VX-PJ)(x_1^2 - x_2^2)\right| + C\,. \tag{5.26}$$

In the second case the corresponding sigma models are torsionless and their metrics are flat. The dilaton field obtained from (3.4) is

$$\Phi(x) = \ln|(1 + G(x_1 - \epsilon x_2))(1 - H(x_1 + \epsilon x_2))| + C\,. \tag{5.27}$$

In both cases the β-function equations (2.8)–(2.10) are satisfied. Note that there is another solution of (2.8)–(2.10) in the case (5.25), namely $\Phi(x) = \text{const.}$ while Φ given by (5.27) is a nontrivial solution of

$$\nabla_i \nabla_j \Phi = 0\,, \quad \nabla^j \Phi \nabla_j \Phi = 0\,, \qquad \forall i,j\,. \tag{5.28}$$

- Decomposition $(5|2.i)$ and

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & Q & \epsilon - 1 \\ X & \epsilon + 1 & 0 \end{pmatrix} \tag{5.29}$$

or

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & (G+2)G/J & G \\ X & G+2 & J \end{pmatrix}. \tag{5.30}$$

The corresponding sigma model are torsionless and their metrics are flat. The dilaton field given by (3.4) is constant.

- Decomposition $(5.ii|6_0)$. Transformations (3.1)–(3.3) give models with

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & Q & \epsilon(V-1)+1 \\ X & -\epsilon(X+1)-1 & 0 \end{pmatrix} \tag{5.31}$$

or

$$\tilde{E}_0 = \begin{pmatrix} P & \frac{-2+G-H+V+HV-X+GX-JY}{J} & V \\ Y & \frac{-2+G-H+GH-PJ+V-X+VX}{J} & G \\ X & H & J \end{pmatrix}. \tag{5.32}$$

In the former case the corresponding sigma models are torsionless and their metrics are flat. The dilaton obtained from (3.4) is

$$\begin{aligned} \Phi(x) &= C + \ln\left|V e^{\epsilon(x_1+x_2)} + (V-1)[e^{-x_2}(\epsilon - 1) - \epsilon]\right| + \\ &\quad + \ln\left|X e^{-\epsilon(x_1+x_2)} - (X+1)[e^{-x_2}(\epsilon+1) - \epsilon]\right| \end{aligned} \tag{5.33}$$

and the β-function equations (2.8)–(2.10) are satisfied. Note that once again there is another solution of (2.8)–(2.10) in this case, namely $\Phi(x) = \text{const}$.

In the latter case the corresponding sigma models have nontrivial torsion and their metrics are not flat. The dilaton obtained from (3.4)

$$\begin{aligned}\ln\Bigg|\frac{\mathrm{e}^{-x_1-x_2}}{2\,J}\Big(&-2+G-H+2\,P\,J-V+X-2\,V\,X+\mathrm{e}^{x_1+x_2}(2-2\,V+2\,X)+\\ &+\mathrm{e}^{2(x_1+x_2)}(-2+G-H-2\,P\,J+V-X+2\,V\,X)-\\ &-2\,\mathrm{e}^{2\,x_1+x_2}(-2+G-H-2\,P\,J+V-X+2\,V\,X)-\\ &-4\,\mathrm{e}^{x_1}\,(P\,J-(-1+V)(1+X))\Big)\Bigg|+C\end{aligned} \tag{5.34}$$

is quite complicated and we are not able to check whether the β-function equations (2.8)–(2.10) are satisfied in general. An example of such model was given in [11] and its β-function is known to vanish on the 1-loop level.

- Decomposition (5|1). There is no sigma model corresponding to this decomposition because $\Pi_u(g) = 0$ in (3.2) and the matrix N is not invertible for any transformation matrix (3.1).

5.3 Sigma models on DD12 starting from (4|1)

Constant dilaton exists for the decomposition (4|1) and

$$E_0 = \begin{pmatrix} p & u & v \\ w & q & 0 \\ z & 0 & 0 \end{pmatrix} \tag{5.35}$$

i.e for the tensor

$$F_{ij}(y) = \begin{pmatrix} p & (u+v\,y_1)\,\mathrm{e}^{-y_1} & v\,\mathrm{e}^{-y_1} \\ (w+z\,y_1)\,\mathrm{e}^{-y_1} & q\,\mathrm{e}^{-2\,y_1} & 0 \\ z\,\mathrm{e}^{-y_1} & 0 & 0 \end{pmatrix}. \tag{5.36}$$

The metric is invertible if and only if $q(v+z) \neq 0$. The corresponding sigma models are torsionless, their metrics are flat and $\Phi^{(0)}(y) = C - 2y_1$.

All these models can be obtained by a plurality transformation from those with

$$E_0 = \kappa \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.37}$$

Once again we can get rid off the overall constant $\kappa \in \mathbb{R}\backslash\{0\}$ by the scaling transformation (3.5).

Beside that by plurality transformations we get models with

- Decompositions $(4|2.i)$, $(4|2.ii)$ and

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & Q & 0 \\ X & 0 & 0 \end{pmatrix} \tag{5.38}$$

JHEP10(2004)045

or

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & Q & -2\epsilon \\ X & 2\epsilon & 0 \end{pmatrix} \tag{5.39}$$

where $\epsilon = 1$ for $(4|2.i)$ and $\epsilon = -1$ for $(4|2.ii)$. The metric is invertible if and only if $Q(V+X) \neq 0$. The corresponding sigma models are torsionless and their metric is flat. The necessary condition for the dilaton transformation is satisfied and the resulting dilaton is constant.

- Decomposition $(2|6_0)$ and

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & Q & \epsilon V \\ X & \epsilon X & 0 \end{pmatrix}. \tag{5.40}$$

The metric is invertible only if $[P + Q - \epsilon(U+Y)](V+X) \neq 0$. The corresponding sigma models are torsionless and their metric is flat so that this model allows constant dilaton in spite of the fact that the necessary condition for the transformation of (constant) dilaton of $(4|1)$ is violated because $R^{13} = \pm 1$.

- Decomposition $(6_0|2)$ and

$$\tilde{E}_0 = \begin{pmatrix} P & \epsilon P & V \\ \epsilon P & P & G \\ X & H & J \end{pmatrix}. \tag{5.41}$$

The metric is invertible if and only if $P[G + H - \epsilon(V+X)] \neq 0$. The corresponding sigma models are torsionless but their metric is not flat. Neither Riemann nor Ricci tensors vanish. Gauss curvature vanishes. The necessary condition for the dilaton transformation is satisfied. The dilaton is

$$\Phi(x) = 2\epsilon x_3 + C$$

and the β-function equations (2.8)–(2.10) hold.

- Decomposition $(4.ii|6_0)$ and

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & Q & V \\ X & X & 0 \end{pmatrix} \tag{5.42}$$

or

$$\tilde{E}_0 = \begin{pmatrix} P & U & 1+W \\ Y & Q & 1-W \\ -1+Z & -1-Z & 0 \end{pmatrix}. \tag{5.43}$$

The corresponding sigma models are torsionless and their metrics are flat so that this is once again a model allowing a constant dilaton in spite of the fact that the necessary condition for the transformation of (constant) dilaton of $(4|1)$ is violated because $R^{11} = R^{12} = \pm 1$.

5.4 Sigma models on DD13 starting from $(3|1)$

Constant dilaton exists for the decomposition $(3|1)$ and

$$E_0 = \begin{pmatrix} p & u & v \\ w & q & -q \\ z & -q & q \end{pmatrix} \tag{5.44}$$

i.e. for the tensor

$$F_{ij}(y) = \begin{pmatrix} p & \frac{u-v+(u+v)\mathrm{e}^{-2y_1}}{2} & \frac{-u+v+(u+v)\mathrm{e}^{-2y_1}}{2} \\ \frac{w-z+(w+z)\mathrm{e}^{-2y_1}}{2} & q & -q \\ \frac{-w+z+(w+z)\mathrm{e}^{-2y_1}}{2} & -q & q \end{pmatrix}. \tag{5.45}$$

The metric is invertible if and only if $q(u+v+w+z) \neq 0$. The corresponding sigma models are torsionless, their metrics are flat.

All these models can be obtained by a plurality transformation from those with

$$E_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & -\epsilon \\ 1 & -\epsilon & \epsilon \end{pmatrix}. \tag{5.46}$$

Beside that by plurality transformations we get models with

- Decompositions $(3|2)$ and

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & J & -J \\ X & -J & J \end{pmatrix} \tag{5.47}$$

or

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & J & -2+J \\ X & 2+J & J \end{pmatrix} \tag{5.48}$$

where $\mathrm{sgn}J = \epsilon$. The metric is invertible if and only if $J(U+V+X+Y) \neq 0$ in the first case and if and only if $J(U-V-X+Y) \neq 0$ in the second. The corresponding sigma models are torsionless and their metrics are flat. The necessary condition for the dilaton transformation is satisfied and the dilaton is constant.

5.5 Sigma models on DD15 starting from $(7_0|1)$

Constant dilaton exists for the decomposition[11] $(7_0|1)$ and

$$E_0 = \begin{pmatrix} p & u & v \\ -u & p & g \\ z & h & r \end{pmatrix} \tag{5.49}$$

i.e. for the tensor

$$F_{ij}(y) = \begin{pmatrix} p & u & v - u\,y_1 + p\,y_2 \\ -u & p & g - p\,y_1 - u\,y_2 \\ z + u\,y_1 + p\,y_2 & h - p\,y_1 + u\,y_2 & r(y) \end{pmatrix} \tag{5.50}$$

[11] This set of models was also investigated in [11].

where

$$r(y) = r - (g+h)\,y_1 + (v+z)\,y_2 + p\,({y_1}^2 + {y_2}^2)\,.$$

The metric is invertible if and only if $p(4pr - (g+h)^2 - (v+z)^2) \neq 0$. The corresponding sigma models are torsionless and their metrics are flat.

The function $\Phi^{(0)}$ is constant in this case so that we have no problems with its possible dependence on coordinates of auxiliary groups $\tilde{G}_u$ and the formula (3.4) is valid for any decomposition.

All these models can be obtained by a plurality transformation and scaling (3.5) from those with

$$E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}. \tag{5.51}$$

Beside that by plurality transformations we get

- Decompositions $(1|7_0)$ and

$$\tilde{E_0} = \begin{pmatrix} (QJ - GH + VX)/J & (HV + GX - JY)/J & V \\ Y & Q & G \\ X & H & J \end{pmatrix}, \tag{5.52}$$

  where models with definite metric are obtained by plurality from (5.51) with + sign, models with indefinite metric from (5.51) with − sign.

  The corresponding sigma models have nontrivial torsion and their metrics are not flat. The dilaton field is

$$\Phi(x) = \ln|1 + (G - H)x_1 + (X - V)x_2 + (QJ - GH)(x_1^2 + x_2^2)| + C \tag{5.53}$$

  and the β-function equations (2.8)–(2.10) are satisfied.

5.6 Sigma models on DD19 starting from $(2|1)$

Constant dilaton exists for the decomposition $(2|1)$ and

$$E_0 = \begin{pmatrix} 0 & u & v \\ w & q & g \\ z & h & r \end{pmatrix} \tag{5.54}$$

i.e. for the tensor

$$F_{ij}(y) = \begin{pmatrix} 0 & u & v \\ w & q & g + w\,y_2 \\ z & h + u\,y_2 & r + (v+z)\,y_2 \end{pmatrix}. \tag{5.55}$$

The corresponding sigma models are torsionless and their metrics are flat, $\Phi^{(0)} = C$.

All these models can be obtained by a plurality transformation from those with

$$E_0 = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.56}$$

Beside that by the plurality transformations we get

- Decompositions $(1|2)$ and

$$\tilde{E}_0 = \begin{pmatrix} P & U & V \\ Y & Q & G \\ X & H & J \end{pmatrix}, \qquad QJ = GH\,. \tag{5.57}$$

For $G = H$ the corresponding sigma models are flat and torsionless, for $G \neq H$ they have nontrivial torsion and their Gauss curvature is not zero, i.e. their metrics are not flat.

The dilaton field

$$\Phi(x) = \ln|1 + (G - H)x_1| + C \tag{5.58}$$

obtained from (3.4) satisfies the vanishing β-function equations (2.8)–(2.10).

5.7 Sigma models on DD22

DD22 denotes the abelian Drinfeld double having just one class of isomorphic Manin triples $(1|1)$. Constant dilaton exists for arbitrary E_0. The tensor F is constant so that all models are flat and torsionless and can be obtained by a plurality transformation from that with

$$E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}. \tag{5.59}$$

6. Conclusions

In the present paper we have provided an explanation of the origin of the dilaton puzzle and given criteria establishing when the new dilaton exists. It became clear that in generic case of $\Phi^{(0)}$ nonconstant (or in general, not a function of spectator fields only) the simplest dual, obtained by the interchange of the subalgebras, doesn't exist. This casts some doubts on the use of name "duality".

We have presented several sets of mutually equivalent Poisson-Lie T-plural models and found several examples of properties not encountered before in the context of Poisson-Lie T-plural models.

Firstly, we know that there are models, namely (5.25), (5.31) that allow two different dilatons, i.e. (5.27), (5.33) respectively, and the constant one. It is quite surprising to have two seemingly distinct solutions of the vanishing β-function equations (2.8)–(2.10) for the same flat metric. The only explanation of (5.28) is that in "flat" coordinates the dilaton is a linear function of coordinates such that its derivative is a null-vector. This knowledge can be helpful if one attempts to find the flat coordinates.

Secondly, in (5.24) and (5.25) (and similarly in (5.57)) we have an example of T-plural models on the same Manin triple with different matrices E_0,[12] some of them being flat with constant dilaton, the others being curved and with nontrivial dilaton.

Thirdly, by different choices of E_0 one can find models inequivalent in T-plurality sense on the same Manin triple with rather different properties — compare e.g. (5.10) and (5.21).

[12] I.e. on isomorphic Manin triples with the same subspaces $\mathcal{E}^{\pm}$, see [11].

JHEP10(2004)045

Of course, in general it is highly unprobable that a dilaton satisfying (2.8)–(2.10) exists for a generic choice of E_0, e.g. general forms (5.10) and (5.21) of E_0 are quite special, fixing in both cases two relations between elements of E_0.

Also it became clear that coordinates based on 1-parameter subgroups which were used in the paper are suitable for description of invariant vector fields, plurality transformation etc., but may be rather inconvenient for understanding geometric properties of the models found. On the other hand the group structure may become rather complicated in coordinates in which geometric properties are transparent, e.g. coordinates such that metric becomes Minkowski (resp. euclidean) in flat cases. We know only two cases when a suitable choice of 1-parameter subgroups leads to explicitly flat coordinates — namely by renaming the basis elements $X_1' = X_3, X_2' = X_2, X_3' = X_1$ and simple linear transformation of coordinates one gets explicitly constant diagonal metric in (5.21) and (5.49) for $g = h = v = z = 0$. Since our current goal was to get better understanding of Poisson-Lie T-plurality through investigation of examples we presented the results in coordinates suited for description of group properties, i.e. via 1-parameter subgroups. In future we plan also to investigate the geometric structure of the models we found, e.g. whether their metric can be simplified in some suitable coordinates or whether they can be related to some models with already known properties. Also it might easily happen that models with different E_0 or even with different group structure can be, as far as metric, torsion potential and dilaton are considered, transformed one into the other by change of coordinates. This is the case e.g. in all flat torsion-free, constant dilaton cases with indefinite metric — they are all (locally) equivalent to the model with Minkowski metric[13]. It is an open question between which non-flat models a similar identification is also possible. On the other hand, the group structures involved may have importance in global aspects which are currently not understood in Poisson-Lie T-plurality setting. One should be also aware that using one fixed transformation (3.1) one may arrive to different models starting from equivalent, e.g. flat, models with different E_0 (as illustrated e.g. in (5.57)).

This investigation would be also related to another question of great practical importance. At present, there is no way of determining whether a given model, i.e. metric together with torsion potential and dilaton prescribed in some coordinates, can be dualized (pluralized). There exist a criterion given in [1], the so-called generalized isometry condition

$$(\mathcal{L}_{v_c} F)_{ij} = \tilde{f}_c^{ab} v_a^m v_b^n F_{im} F_{nj} \tag{6.1}$$

but it assumes that the group structure is already known, i.e. that left-invariant vector fields v_a^m are given. In the contrary to ordinary isometries with $\tilde{f}_c^{ab} = 0$, generalized isometries satisfying (6.1) cannot be investigated one by one, they must be considered as a set due to their nontrivial interplay on the right-hand side of (6.1). At the present there is no algorithm that would indicate which group one should consider and how it should be expressed in initially given coordinates. The best algorithm available at the moment seems to be to find first the algebra of ordinary isometries, i.e. Killing vectors, and the corresponding

[13]Note that metrics (5.49) and (5.59) allow also non-equivalent euclidean signature, which is probably physically less significant.

JHEP10(2004)045

group, then to determine its subgroups acting freely on the target manifold. For each of those subgroups G one should then parametrize their orbits by spectator coordinates (if the action is not transitive) and to determine the Drinfeld double containing Manin triple $(\mathcal{G}|1)$. Then one can investigate models given by other decomposition of that Drinfeld double provided new metric (3.2) and new dilaton (3.4) exists as explained in section 4 and [11]. We hope that in future methods or clues for determination of suitable algebras of generalized isometries of a given model will be found and that it will be possible to find truly Poisson-Lie T-plural models even for physically interesting metrics with few or without ordinary isometries.

As mentioned before, it is rather surprising that in all known examples it is simultaneously either possible or impossible to find new dilatons in all models obtained by moving subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$ in the Drinfeld double without altering their structures (i.e. transformations to isomorphic Manin triples); such transformation can be also interpreted as a certain change of the matrix E_0 in the given model (2.2). This coincidence seems to indicate a deeper relation between conformal invariance and the Poisson-Lie T-plurality and provides additional motivation for its further study.

The curvatures of the models defined by (5.15), (5.52) and (5.57) diverge[14] on hypersurfaces where the corresponding dilatons are also divergent (i.e. behave like $\ln(0)$). The singular hypersurfaces are parametrized in coordinate space as hyperbolic cylinder, elliptic cylinder and hyperplane respectively. Nevertheless, the metric, torsion potential and dilaton appear to have a reasonable continuation behind the singularity since all of them are well-defined there by formulae (3.2), (3.4). We don't know at the moment whether such backgrounds have meaningful physical interpretation, e.g. as branes.

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A. Drinfeld doubles

The Drinfeld double D is defined as a connected Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad-invariant nondegenerate bilinear form $\langle\, .\, ,.\, \rangle$ can be decomposed into a pair of subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ maximally isotropic with respect to $\langle\, .\, ,.\, \rangle$. The dimensions of the subalgebras are equal and bases $\{X_i\}, \{\tilde{X}^i\}$ in the subalgebras can be chosen so that

$$\langle X_i, X_j\rangle = 0,\ \langle X_i, \tilde{X}^j\rangle = \langle \tilde{X}^j, X_i\rangle = \delta_i^j,\ \langle \tilde{X}^i, \tilde{X}^j\rangle = 0. \tag{A.1}$$

We shall assume that any basis of Manin triple considered in this paper satisfies (A.1).

[14] In the cases (5.15), (5.52) this was observed already in [11].

Due to the ad-invariance of $\langle\, .\, , .\,\rangle$ the structure constants of $\mathcal{D}$ are determined by the structure of its maximally isotropic subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$, i.e. if in bases $\{X_i\}, \{\tilde{X}^i\}$ the Lie products are given by

$$[X_i, X_j] = f_{ij}{}^k X_k,\ [\tilde{X}^i, \tilde{X}^j] = \tilde{f^{ij}}_k \tilde{X}^k$$

then

$$[X_i, \tilde{X}^j] = f_{ki}{}^j \tilde{X}^k + \tilde{f^{jk}}_i X_k\,. \tag{A.2}$$

For given Drinfeld double several Manin triples may exist, i.e.

$$(\mathcal{G}|\tilde{\mathcal{G}}) \cong (\tilde{\mathcal{G}}|\mathcal{G}) \cong (\mathcal{G}'|\tilde{\mathcal{G}}') \cong \ldots$$

Examples of transformations between $(\mathcal{G}|\tilde{\mathcal{G}})$ and $(\mathcal{G}'|\tilde{\mathcal{G}}')$ are given in [12]. Their general forms are too extensive to display, nevertheless, they were used throughout this paper.

Classification of real six-dimensional Drinfeld doubles and their decompositions into nonisomorphic Manin triples are given in [12]. Here we shall present only those occuring in this paper. Since only the subalgebras denoted in Bianchi classification (see [17] or [18]) by $\mathbf{9}, \mathbf{8}, \mathbf{7_0}, \mathbf{6_0}, \mathbf{2}, \mathbf{1}$ are traceless, we present in each Drinfeld double only Manin triples where at least one of the components has the structure of $\mathbf{7_0}, \mathbf{6_0}, \mathbf{2}, \mathbf{1}$.

A.1 Structure of DD11

$(6_0|1) \cong (6_0|5.ii) \cong (5|1) \cong (5|2.i)$, and dual Manin triples ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$).

$(\mathbf{6_0}|\mathbf{1})$:

$$\begin{aligned} &[X_1, X_2] = 0\,, && [X_2, X_3] = X_1\,, && [X_3, X_1] = -X_2\,, \\ &[\tilde{X}^1, \tilde{X}^2] = 0\,, && [\tilde{X}^2, \tilde{X}^3] = 0\,, && [\tilde{X}^3, \tilde{X}^1] = 0\,. \end{aligned}$$

$(\mathbf{6_0}|\mathbf{5.ii})$:

$$\begin{aligned} &[X_1, X_2] = 0\,, && [X_2, X_3] = X_1\,, && [X_3, X_1] = -X_2\,, \\ &[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^1 + \tilde{X}^2\,, && [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3\,, && [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^3\,. \end{aligned}$$

$(\mathbf{5}|\mathbf{1})$:

$$\begin{aligned} &[X_1, X_2] = -X_2\,, && [X_2, X_3] = 0\,, && [X_3, X_1] = X_3\,, \\ &[\tilde{X}^1, \tilde{X}^2] = 0\,, && [\tilde{X}^2, \tilde{X}^3] = 0\,, && [\tilde{X}^3, \tilde{X}^1] = 0. \end{aligned}$$

$(\mathbf{5}|\mathbf{2.i})$:

$$\begin{aligned} &[X_1, X_2] = -X_2\,, && [X_2, X_3] = 0\,, && [X_3, X_1] = X_3\,, \\ &[\tilde{X}^1, \tilde{X}^2] = 0\,, && [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1\,, && [\tilde{X}^3, \tilde{X}^1] = 0\,. \end{aligned}$$

JHEP10(2004)045

A.2 Structure of DD12

$(6_0|2) \cong (6_0|4.ii) \cong (4|1) \cong (4|2.i) \cong (4|2.ii)$, and dual Manin triples.

$(\mathbf{6_0}|\mathbf{2})$:

$$[X_1, X_2] = 0\,, \qquad [X_2, X_3] = X_1\,, \qquad [X_3, X_1] = -X_2,$$
$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3\,, \qquad [\tilde{X}^2, \tilde{X}^3] = 0\,, \qquad [\tilde{X}^3, \tilde{X}^1] = 0\,.$$

$(\mathbf{6_0}|\mathbf{4.ii})$:

$$[X_1, X_2] = 0\,, \qquad [X_2, X_3] = X_1\,, \qquad [X_3, X_1] = -X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = (-\tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3)\,, \qquad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3\,, \qquad [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^3.$$

$(\mathbf{4}|\mathbf{1})$:

$$[X_1, X_2] = -X_2 + X_3\,, \qquad [X_2, X_3] = 0\,, \qquad [X_3, X_1] = X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0\,, \qquad [\tilde{X}^2, \tilde{X}^3] = 0\,, \qquad [\tilde{X}^3, \tilde{X}^1] = 0.$$

$(\mathbf{4}|\mathbf{2.i})$:

$$[X_1, X_2] = -X_2 + X_3\,, \qquad [X_2, X_3] = 0\,, \qquad [X_3, X_1] = X_3,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0\,, \qquad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1\,, \qquad [\tilde{X}^3, \tilde{X}^1] = 0\,.$$

$(\mathbf{4}|\mathbf{2.ii})$:

$$[X_1, X_2] = -X_2 + X_3\,, \qquad [X_2, X_3] = 0\,, \qquad [X_3, X_1] = X_3,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0\,, \qquad [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^1\,, \qquad [\tilde{X}^3, \tilde{X}^1] = 0\,.$$

A.3 Structure of DD13

$(3|1) \cong (3|2) \cong (3|3.ii) \cong (3|3.iii)$, and dual Manin triples.

$(\mathbf{3}|\mathbf{1})$:

$$[X_1, X_2] = -X_2 - X_3\,, \qquad [X_2, X_3] = 0\,, \qquad [X_3, X_1] = X_2 + X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0\,, \qquad [\tilde{X}^2, \tilde{X}^3] = 0\,, \qquad [\tilde{X}^3, \tilde{X}^1] = 0.$$

$(\mathbf{3}|\mathbf{2})$:

$$[X_1, X_2] = -X_2 - X_3\,, \qquad [X_2, X_3] = 0\,, \qquad [X_3, X_1] = X_2 + X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0\,, \qquad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1\,, \qquad [\tilde{X}^3, \tilde{X}^1] = 0.$$

A.4 Structure of DD15

$(7_0|1) \cong (1|7_0)$.

$(\mathbf{7_0}|\mathbf{1})$:

$$[X_1, X_2] = 0\,, \qquad [X_2, X_3] = X_1\,, \qquad [X_3, X_1] = X_2,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0\,, \qquad [\tilde{X}^2, \tilde{X}^3] = 0\,, \qquad [\tilde{X}^3, \tilde{X}^1] = 0.$$

JHEP10(2004)045

A.5 Structure of DD19

$(2|1) \cong (1|2)$.

(2|1):

$$[X_1, X_2] = 0\,, \qquad [X_2, X_3] = X_1\,, \qquad [X_3, X_1] = 0,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0\,, \qquad [\tilde{X}^2, \tilde{X}^3] = 0\,, \qquad [\tilde{X}^3, \tilde{X}^1] = 0.$$

A.6 Structure of DD22

(1|1):

$$[X_i, X_j] = 0\,, \qquad [\tilde{X}^i, \tilde{X}^j] = 0\,.$$

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